

BOREL EQUIVALENCE RELATIONS BETWEEN ℓ_1 AND ℓ_p

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ABSTRACT. In this paper, we show that, for each $p > 1$, there are continuum many Borel equivalence relations between \mathbb{R}^ω/ℓ_1 and \mathbb{R}^ω/ℓ_p ordered by \leq_B which are pairwise Borel incomparable.

1. INTRODUCTION

A *Polish space* is a topological space that admits a compatible complete separable metric. For more details in descriptive set theory, one can see [4]. Let X, Y be Polish spaces, E, F equivalence relations on X, Y respectively, we say E is *Borel reducible* to F , denoted by $E \leq_B F$, if there exists a Borel function $\theta : X \rightarrow Y$ satisfying

$$xE\hat{x} \iff \theta(x)F\theta(\hat{x}).$$

We say E is *strictly Borel reducible* to F , $E <_B F$ in notation, if $E \leq_B F$ but $F \not\leq_B E$. We refer to [3] for background on Borel reducibility.

R. Dougherty and G. Hjorth [1] proved that, for $p, q \geq 1$,

$$\mathbb{R}^\omega/\ell_p \leq_B \mathbb{R}^\omega/\ell_q \iff p \leq q.$$

A question of S. Gao in [2] asking whether \mathbb{R}^ω/ℓ_p is the greatest lowest bound of $\{\mathbb{R}^\omega/\ell_q : p < q\}$. T. Mátrai answer this question in the negative by showing, for $1 \leq p < q$, every linear order which embeds into $(P(\omega)/\text{fin}, \subset)$ also embeds into the set of equivalence relations between \mathbb{R}^ω/ℓ_p and \mathbb{R}^ω/ℓ_q ordered by $<_B$ (see [5], Corollary 31).

We can see that all equivalence relations considered in Mátrai's paper [5] are pairwise Borel comparable. A question arises naturally that, for $1 \leq p < q$, whether there are equivalence relations E, F such that $\mathbb{R}^\omega/\ell_p \leq_B E, F \leq_B \mathbb{R}^\omega/\ell_q$ but E, F are incomparable. Both Gao and Mátrai asked this question in the special case $p = 1, q = 2$. In this paper, we show that, for each $p > 1$, there are continuum many pairwise Borel incomparable equivalence relations between \mathbb{R}^ω/ℓ_1 and \mathbb{R}^ω/ℓ_p .

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2. SOME NOTES ON \mathbf{E}_f

We denote by \mathbb{R}^+ the set of nonnegative real numbers. Let $f : [0, 1] \rightarrow \mathbb{R}^+$. Mátrai [5] defined the relation \mathbf{E}_f on $[0, 1]^\omega$ by setting, for every $(x_n)_{n < \omega}, (y_n)_{n < \omega} \in [0, 1]^\omega$,

$$(x_n)\mathbf{E}_f(y_n) \iff \sum_{n < \omega} f(|y_n - x_n|) < \infty.$$

It is straightforward that \mathbf{E}_f is a Borel relation whenever f is Borel.

The following proposition answers when \mathbf{E}_f is an equivalence relation.

Proposition 2.1 (Mátrai [5], Proposition 2). *Let $f : [0, 1] \rightarrow \mathbb{R}^+$ be a bounded function. Then \mathbf{E}_f is an equivalence relation iff the following conditions hold:*

$$(R_1) \ f(0) = 0;$$

$$(R_2) \ \text{there is a } C \geq 1 \text{ such that for every } x, y \in [0, 1] \text{ with } x + y \in [0, 1],$$

$$f(x + y) \leq C(f(x) + f(y)),$$

$$f(x) \leq C(f(x + y) + f(y)).$$

A nonreducibility result was obtained in [5] for a class of \mathbf{E}_f 's as follows.

Theorem 2.2 (Mátrai [5], Theorem 18). *Let $1 \leq \alpha < \infty$ and let $\varphi, \psi : [0, 1] \rightarrow [0, +\infty)$ be continuous. Set $f(x) = x^\alpha \varphi(x), g(x) = x^\alpha \psi(x)$ for $x \in [0, 1]$ and suppose that f, g are bounded and \mathbf{E}_f and \mathbf{E}_g are equivalence relations. Suppose $\psi(x) > 0$ ($x > 0$), and*

$$(A_1) \ \text{there exist } \varepsilon > 0, M < \omega \text{ such that for every } n > M \text{ and } x, y \in [0, 1],$$

$$\varphi(x) \leq \varepsilon \varphi(y) \varphi(1/2^n) \Rightarrow x \leq \frac{y}{2^{n+1}};$$

$$(A_2) \ \lim_{n \rightarrow \infty} \psi(1/2^n) / \varphi(1/2^n) = 0.$$

Then $\mathbf{E}_g \not\leq_B \mathbf{E}_f$.

Remark 2.3. We may replace condition (A_2) in the theorem by

$$(A_2)' \ \liminf_{n \rightarrow \infty} \psi(1/2^n) / \varphi(1/2^n) = 0.$$

In fact, we can check that the proof for Theorem 18 of [5] is still valid under condition $(A_2)'$. In this paper, condition $(A_2)'$ is the key to prove incomparability between equivalence relations.

Mostly, we focus on equivalence relations \mathbf{E}_f in which $f(x) = x^\alpha \varphi(x)$ for $x \in [0, 1]$ with φ continuous increasing.

Lemma 2.4. *Let $\alpha \geq 1$ and $\varphi : [0, 1] \rightarrow [0, \infty)$ be an increasing function with $\varphi(1/2) > 0$. Set $f(x) = x^\alpha \varphi(x)$ for $x \in [0, 1]$. If there exists $\delta > 0$ such that, for each $n > 1$,*

$$\varphi(1/2^n) \geq \max_{1 \leq i \leq n-1} \{\delta \varphi(1/2^i) \varphi(1/2^{n-i})\},$$

then \mathbf{E}_f is an equivalence relation and condition (A_1) in Theorem 2.2 holds.

Proof. Note that for $n > 1$ we have $\varphi(1/2^n) \geq \delta\varphi(1/2)\varphi(1/2^{n-1})$. Since $\varphi(1/2) > 0$ and φ is increasing, we have $\varphi(x) > 0$ for $x > 0$. By Proposition 2.1 and Theorem 2.2, we need only to check (R_1) , (R_2) , and (A_1) .

For (R_1) , $f(0) = 0$ is trivial.

For (R_2) , let $x, y \in [0, 1]$ with $x + y \in [0, 1]$. Without loss of generality, we can assume that $x \geq y > 0$. Since $f(x) = x^\alpha \varphi(x)$ is increasing, we have

$$f(x) \leq f(x + y) \leq (f(x + y) + f(y)).$$

If $x > 1/4$, then

$$f(x + y) \leq f(1) = \frac{4^\alpha \varphi(1)}{\varphi(1/4)} f(1/4) \leq \frac{4^\alpha \varphi(1)}{\varphi(1/4)} (f(x) + f(y)).$$

If $x \leq 1/4$, let $x \in (1/2^{n+1}, 1/2^n]$ with $n > 1$. Then

$$f(x + y) \leq f(2x) \leq f(1/2^{n-1}) = \frac{1}{2^{(n-1)\alpha}} \varphi(1/2^{n-1}),$$

$$f(x) \geq f(1/2^{n+1}) = \frac{1}{2^{(n+1)\alpha}} \varphi(1/2^{n+1}) \geq \frac{\delta}{2^{(n+1)\alpha}} \varphi(1/4) \varphi(1/2^{n-1}).$$

Thus we have

$$f(x + y) \leq \frac{4^\alpha}{\delta \varphi(1/4)} (f(x) + f(y)).$$

Therefore, $C = \max \left\{ 1, \frac{4^\alpha \varphi(1)}{\varphi(1/4)}, \frac{4^\alpha}{\delta \varphi(1/4)} \right\}$ witnesses that (R_2) holds.

For (A_1) , fix a $0 < \varepsilon < \min\{1/\varphi(1), \delta\varphi(1/4)/\varphi(1), \delta^2\varphi(1/4)\}$. For $x, y \in [0, 1]$ and $n > 0$, assume for contradiction that

$$\varphi(x) \leq \varepsilon \varphi(y) \varphi(1/2^n), \text{ but } x > \frac{y}{2^{n+1}}.$$

If $y = 0$, since $\varphi(x) \leq \varepsilon \varphi(0) \varphi(1/2^n) \leq \varphi(0)$, we have $x = 0$. It contradicts to $x > \frac{y}{2^{n+1}}$.

If $y > 0$, let $y \in (1/2^{m+1}, 1/2^m]$ for some $m \in \omega$, then $x > 1/2^{m+n+2}$. If $m = 0$, we have

$$\varepsilon \varphi(1) \varphi(1/2^n) \geq \varphi(x) \geq \varphi(1/2^{n+2}) \geq \delta \varphi(1/4) \varphi(1/2^n),$$

contradicting $\varepsilon < \delta \varphi(1/4)/\varphi(1)$. If $m \geq 1$, we have

$$\begin{aligned} \varepsilon \varphi(1/2^m) \varphi(1/2^n) &\geq \varphi(x) \geq \varphi(1/2^{m+n+2}) \\ &\geq \delta \varphi(1/2^{m+2}) \varphi(1/2^n) \\ &\geq \delta^2 \varphi(1/4) \varphi(1/2^m) \varphi(1/2^n), \end{aligned}$$

contradicting $\varepsilon < \delta^2 \varphi(1/4)$. □

3. PAIRWISE INCOMPARABLE EQUIVALENCE RELATIONS

From Lemma 2.4, we can define φ from a decreasing sequence $(u_n)_{n < \omega}$ by setting $\varphi(1/2^n) = u_n$ and then extend φ to $[0, 1]$ to be a continuous increasing function which is affine on each $[1/2^{n+1}, 1/2^n]$.

Lemma 3.1. *Let $0 < \delta, \lambda < 1$ and $u_0 = u_1 = 1$. For $n > 1$, suppose that $u_n = u_{n-1}$ or $u_n = \lambda u_{n-1} + (1 - \lambda) \max_{1 \leq i \leq n-1} \{\delta u_i u_{n-i}\}$. Then we have, for each $n > 1$,*

$$u_{n-1} \geq u_n \geq \max_{1 \leq i \leq n-1} \{\delta u_i u_{n-i}\}.$$

Proof. We argue by induction on n . If $n = 2$, then $u_2 = u_1$ or $u_2 = \lambda u_1 + (1 - \lambda) \delta u_1^2$. So $u_1 \geq u_2 \geq \delta u_1^2$.

For $n > 2$, by induction hypothesis, $u_{k-1} \geq u_k \geq \max_{1 \leq i \leq k-1} \{\delta u_i u_{k-i}\}$ for each $2 \leq k < n$. Thus

$$u_{n-1} \geq \max_{1 \leq i \leq n-2} \{\delta u_i u_{n-i-1}\} \geq \max_{1 \leq i \leq n-2} \{\delta u_i u_{n-i}\}.$$

Note that $u_{n-1} \geq \delta u_{n-1} u_1$, we have $u_{n-1} \geq \max_{1 \leq i \leq n-1} \{\delta u_i u_{n-i}\}$. Then by the definition of u_n ,

$$u_{n-1} \geq u_n \geq \max_{1 \leq i \leq n-1} \{\delta u_i u_{n-i}\}.$$

□

Lemma 3.2. *Let $\beta > \alpha \geq 1$, $0 < \delta < 1$ and $\lambda = 2^{\alpha-\beta}$. Suppose that $(u_n)_{n < \omega}$ is a sequence as in Lemma 3.1 and $\varphi : [0, 1] \rightarrow [0, \infty)$ is a continuous increasing function with $\varphi(1/2^n) = u_n$ for each $n < \omega$. Set $f(x) = x^\alpha \varphi(x)$ for $x \in [0, 1]$. Then \mathbf{E}_f is an equivalence relation and*

$$\mathbb{R}^\omega / \ell_1 \leq_B \mathbf{E}_f \leq_B \mathbb{R}^\omega / \ell_\beta.$$

Proof. (1) From Lemma 3.1, we have

$$\varphi(1/2^n) = u_n \geq \max_{1 \leq i \leq n-1} \{\delta u_i u_{n-i}\} = \max_{1 \leq i \leq n-1} \{\delta \varphi(1/2^i) \varphi(1/2^{n-i})\}.$$

Thus by Lemma 2.4, \mathbf{E}_f is an equivalence relation.

(2) Fix a bijection $\langle \cdot, \cdot, \cdot \rangle : \{0, 1\} \times \omega \times \omega \rightarrow \omega$. For each $n \in \omega$, find a $c_n \in [0, 1]$ such that $0 < f(c_n) < 1/2^n$. We define $\theta_1 : \mathbb{R}^\omega \rightarrow [0, 1]^\omega$ by, for $(x_n)_{n < \omega} \in \mathbb{R}^\omega$, setting $\theta_1((x_n)_{n < \omega}) = (y_m)_{m < \omega}$ with

$$\begin{aligned} y_m = c_n &\iff m = \langle 0, n, k \rangle, x_n \geq 0, k < [x_n / f(c_n)], \\ &\text{or } m = \langle 1, n, k \rangle, x_n < 0, k < [-x_n / f(c_n)], \end{aligned}$$

and $y_m = 0$ otherwise. It is easy to check that θ_1 is Borel. For $(x_n)_{n < \omega}, (\hat{x}_n)_{n < \omega} \in \mathbb{R}^\omega$, if $\theta_1((x_n)_{n < \omega}) = (y_m)_{m < \omega}, \theta_1((\hat{x}_n)_{n < \omega}) = (\hat{y}_m)_{m < \omega}$, we have

$$|x_n - \hat{x}_n| - 1/2^{n-1} < \sum f(|y_m - \hat{y}_m|) < |x_n - \hat{x}_n| + 1/2^{n-1},$$

where \sum ranges over $\{m = \langle i, k, n \rangle : y_m \neq \hat{y}_m, k < \omega, i = 0, 1\}$. Thus

$$\sum_{n < \omega} |x_n - \hat{x}_n| < \infty \iff \sum_{m < \omega} f(|y_m - \hat{y}_m|) < \infty.$$

Therefore, θ_1 witnesses that $\mathbb{R}^\omega / \ell_1 \leq_B \mathbf{E}_f$.

(3) For proving $\mathbf{E}_f \leq_B \mathbb{R}^\omega / \ell_\beta$, by Theorem 16 of [5], we need only to find a function $\kappa : \{1/2^i : i < \omega\} \rightarrow [0, 1]$ and $L \geq 1$ satisfying that, for each $n < \omega$,

$$(i) \quad f(1/2^n) = \sum_{i=0}^n (\kappa(1/2^i) / 2^{n-i})^\beta;$$

- (ii) $\sum_{i=n}^{\infty} \kappa(1/2^i)^\beta \leq L \sum_{i=0}^n (\kappa(1/2^i)/2^{n-i})^\beta$;
- (iii) $\kappa(1/2^n) \leq L \cdot \max\{\kappa(1/2^i)/2^{n-i} : i < n\}$.

To satisfy (i), we shall let $\kappa(1) = f(1) = u_0 = 1$ and, for $n > 0$,

$$\kappa(1/2^n)^\beta = f(1/2^n) - f(1/2^{n-1})/2^\beta = (u_n - \lambda u_{n-1})/2^{n\alpha}.$$

Note that $u_1 - \lambda u_0 = 1 - \lambda \in [0, 1]$ and, for $n > 1$,

$$(1 - \lambda)u_{n-1} \geq u_n - \lambda u_{n-1} \geq (1 - \lambda) \max_{1 \leq i \leq n-1} \{\delta u_i u_{n-i}\},$$

so $u_n - \lambda u_{n-1} \in [0, 1]$. We see that $\kappa(1/2^n)$ is well defined.

Let $L = \max\{\sum_{k=0}^{\infty} 2^{-k\alpha}, 2, (\delta 2^\alpha)^{-1/\beta}\}$.

By the definition of κ , we have $\kappa(1/2^n)^\beta \leq f(1/2^n) = \varphi(1/2^n)/2^{n\alpha}$.

Hence

$$\sum_{i=n}^{\infty} \kappa(1/2^i)^\beta \leq \sum_{i=n}^{\infty} \varphi(1/2^i)/2^{i\alpha} \leq \sum_{i=n}^{\infty} \varphi(1/2^n)/2^{i\alpha} = f(1/2^n) \sum_{i=n}^{\infty} \frac{1}{2^{(i-n)\alpha}}.$$

From (i), we know (ii) is satisfied.

For (iii), if $n = 1$, then $\kappa(1/2) \leq 1 \leq L\kappa(1)/2$.

Note that for each $n > 1$, we have

$$\begin{aligned} \kappa(1/2^n)^\beta &= (u_n - \lambda u_{n-1})/2^{n\alpha} \leq (1 - \lambda)u_{n-1}/2^{n\alpha} \\ &\leq (1 - \lambda)u_{n-2}/2^{n\alpha} \leq (1 - \lambda) \max_{1 \leq i \leq n-2} \{\delta u_i u_{n-i}\}/(\delta 2^{n\alpha}) \\ &\leq (u_{n-1} - \lambda u_{n-2})/(\delta 2^{n\alpha}) \\ &= \kappa(1/2^{n-1})^\beta/(\delta 2^\alpha). \end{aligned}$$

Then (iii) follows from $\kappa(1/2^n) \leq L\kappa(1/2^{n-1})/2$. □

Theorem 3.3. *For any $\beta > 1$, there is a set of continuous function*

$$\{f_\xi : [0, 1] \rightarrow \mathbb{R}^+ : \xi \in \{0, 1\}^\omega\}$$

such that each \mathbf{E}_{f_ξ} is equivalence relation with $\mathbb{R}^\omega/\ell_1 \leq_B \mathbf{E}_{f_\xi} \leq_B \mathbb{R}^\omega/\ell_\beta$, and for and distinct $\xi, \zeta \in \{0, 1\}^\omega$, we have \mathbf{E}_{f_ξ} and \mathbf{E}_{f_ζ} are Borel incomparable.

Proof. Fix a $0 < \delta < 1$ and a $1 \leq \alpha < \beta$. Let $\lambda = 2^{\alpha-\beta}$. For $s \in \{0, 1\}^{<\omega}$, we denote by $\text{lh}(s)$ the length of s . We are going to construct a finite decreasing sequence $w_s \in [0, 1]^{<\omega}$, a natural number $n_s < \omega$ for every $s \in \{0, 1\}^{<\omega}$, and a sequence of natural numbers $k_0 < k_1 < k_2 < \dots$, satisfying the following list of requirements.

- (a) If $\text{lh}(s) = l$, then $\text{lh}(w_s) = k_l$.
- (b) If $t|l = s$, then $w_t|k_l = w_s$.
- (c) If $\text{lh}(s) = \text{lh}(t) = l$, $s \neq t$, then $k_{l-1} \leq n_s < k_l$ and

$$w_s(n_s)/w_t(n_s) < 1/2^l.$$

Construct by induction on $\text{lh}(s)$. Firstly, let $k_0 = 2$, $w_\emptyset(0) = w_\emptyset(1) = 1$ and $n_\emptyset = 1$. Assume that $k_0 < k_1 < \dots < k_{l-1}$ and for all $\text{lh}(s) < l$, w_s, n_s have been defined. For $\text{lh}(s) = l$ and $n < k_{l-1}$, set $w_s(n) = w_{s|(l-1)}(n)$.

We enumerate $\{0, 1\}^l$ by s_1, s_2, \dots, s_M ($M = 2^l$). Let n_{s_1} be a sufficiently large number specified later, for $s \in \{0, 1\}^l, k_{l-1} \leq n \leq n_{s_1}$, we define

$$w_s(n) = \begin{cases} \lambda w_s(n-1) + (1-\lambda) \max_{1 \leq i \leq n-1} \{\delta w_s(i) w_s(n-i)\}, & s = s_1, \\ w_s(n-1), & s \neq s_1. \end{cases}$$

From Lemma 3.1, we see that w_s is decreasing. Note that $w_{s_1}(i) w_{s_1}(2n-i) \leq w_{s_1}(n)$ for $1 \leq i \leq 2n-1$, we have

$$w_{s_1}(2n) \leq \lambda w_{s_1}(n) + (1-\lambda) \delta w_{s_1}(n) = \delta' w_{s_1}(n),$$

in which $\delta' = \lambda + (1-\lambda)\delta < 1$. Hence $w_{s_1}(2^m n) \leq (\delta')^m w_{s_1}(n) \rightarrow 0$ ($m \rightarrow \infty$). We can find a sufficient large n_{s_1} such that, for $s_1 \neq s \in \{0, 1\}^l$,

$$w_{s_1}(n_{s_1})/w_s(n_{s_1}) < 1/2^l.$$

Follow the same method, we can find $n_{s_1} < n_{s_2} < \dots < n_{s_M}$ such that, for $j = 2, \dots, M$ and $n_{s_{j-1}} < n \leq n_{s_j}$,

$$w_s(n) = \begin{cases} \lambda w_s(n-1) + (1-\lambda) \max_{1 \leq i \leq n-1} \{\delta w_s(i) w_s(n-i)\}, & s = s_j, \\ w_s(n-1), & s \neq s_j. \end{cases}$$

Furthermore, for $s_j \neq s \in \{0, 1\}^l$ we have

$$w_{s_j}(n_{s_j})/w_s(n_{s_j}) < 1/2^l.$$

Letting $k_l = n_{s_M} + 1$, we finish the construction at level l .

For every $\xi \in \{0, 1\}^\omega$, we fix a continuous increasing function $\varphi_\xi : [0, 1] \rightarrow \mathbb{R}^+$ such that $\varphi_\xi(1/2^n) = w_{\xi|l}(n)$ for $l < \omega, n < k_l$. Define $f_\xi(x) = x^\alpha \varphi_\xi(x)$ for $x \in [0, 1]$. From Lemma 3.2, \mathbf{E}_{f_ξ} is equivalence relation, and

$$\mathbb{R}^\omega/\ell_1 \leq_B \mathbf{E}_{f_\xi} \leq_B \mathbb{R}^\omega/\ell_\beta.$$

By Lemma 2.4, condition (A₁) in Theorem 2.2 holds for every φ_ξ . If $\xi \neq \zeta$, then there exists m such that $\xi(m) \neq \zeta(m)$. Let $l > m, s = \zeta|l, t = \xi|l$, we have $s \neq t$. Then

$$\varphi_\zeta(1/2^{n_s})/\varphi_\xi(1/2^{n_s}) = w_s(n_s)/w_t(n_s) < 1/2^l.$$

We see that condition (A₂)' holds. By Remark 2.3, we have $\mathbf{E}_{f_\zeta} \not\leq_B \mathbf{E}_{f_\xi}$. \square

Remark 3.4. Let $1 < \alpha < \beta$, we do not know whether there exist Borel functions $f, g : [0, 1] \rightarrow \mathbb{R}^+$ such that $\mathbf{E}_f, \mathbf{E}_g$ are Borel incomparable equivalence relations with $\mathbb{R}^\omega/\ell_\alpha \leq_B \mathbf{E}_f, \mathbf{E}_g \leq_B \mathbb{R}^\omega/\ell_\beta$.

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